FILLING THE PLANE WITH CONGRUENT CONVEX HEXAGONS WITHOUT OVERLAPPING

By

BÉLA BOLLOBÁS

Department of Geometry of the Eötvös Loránd University, Budapest

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Filling the plane with congruent figures (without overlapping and interstices)[1] seems to be a difficult unsolved problem of geometry. Not even the special case, where the congruent figures are convex polygons, has been settled; there are only partial results.[2]

In the present paper we shall discuss only the filling of the Euclidean plane with congruent convex hexagons, the so called convex hexagon-patterns. Only one of our results should be mentioned here, a corollary of Theorem 1: if the plane can be filled with congruent hexagons, then it can be filled with the same congruent hexagons in such a way that the graph of covering will be isomorphic to the graph of regular tesselation with hexagons.

DEFINITIONS. The term *pattern* will be used for any arrangement of polygons filling the plane.[3] A *tesselation* is a pattern in which no vertex of a polygon is the interior point of a side of another polygon.[4]

A *node* is a point of the plane coinciding with a vertex (consequently at least two vertices are in a node). A node will be called *first-rate*, if it is interior of a side of a polygon, *second-rate* otherwise.

THEOREM 1. *In a convex hexagon-pattern there is a square area of arbitrary side such that each interior node of this square is second-rate and it is surrounded by three hexagons.*

PROOF. Let a denote the area of the hexagon, and let $\frac{d}{2}$ denote the diameter of a circle, which contains the hexagon in its interior.

Let S be a square of side s on the plane of pattern ($s \geqq 3d$) and draw parallels to the sides of S of distance $\frac{d}{2}$. These parallels determine some squares, two of which, S_1 and S_2 have the side $s-d$ resp. $s+d$.

Let n_1 be the number of first-rate nodes in the square S, n_2 the number of interior second-rate nodes of the square surrounded by three hexagons, n_3 by at least four hexagons. Let L denote the number of hexagons having interior points in the square S, and K the number of hexagons contained by the interior of square S (Fig. 1.)

[1] These conditions will not be mentioned further.

[2] See e. g. Б. Н. Делоне [2], F. HAAG [3], [4] and F. LAVES [5].

[3] I. e. each point of the plane is inside or on the boundary of a polygon, but no two polygons have a common interior point. See e. g. COXETER [1].

[4] See e. g. J. MOLNÁR [6].

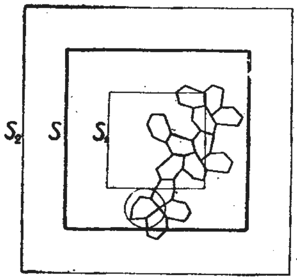

Fig. 1.

Since the square S_2 covers the hexagons, having interior points in S; the sum of the areas of these hexagons is smaller than the area of S_2; consequently

$$L < \frac{(s+d)^2}{a}. \tag{1}$$

We obtain similarly the inequality

$$K > \frac{(s-d)^2}{a}. \tag{2}$$

For the case, when $e = n_1 + n_3$ is limited for any square S, the theorem is trivial, so we may now suppose $e \geqq 5$. We shall show that to prove the the theorem it is sufficient to establish that the quotient $\frac{n_2}{n_1+n_3}$ will exceed an arbitrary number N.

a) If $\frac{n_2}{n_1+n_3} > N$, the square S contains at least $N(n_1+n_3) = Ne$ second-rate nodes, sorrounded by three hexagons. Being $e \geqq 5$, a positive integer f can be found satisfying $e < f^2 \leqq 2e$. Let us divide the square S into f^2 congruent squares. Since $f^2 > e$, at least one of these squares will contain in its interior only second-rate nodes, surrounded by three hexagons. We shall show that the side of this square will exceed any length, if N is sufficiently great.

For the area of this square we have the inequality:

$$\frac{s^2}{f^2} \geqq \frac{s^2}{2e}. \tag{3}$$

The square S contains in its interior at least n_2 nodes, each of them being surrounded by three hexagons, consequently in these nodes there are $3n_2$ vertices and this number can not be greater than $6L$ (the number of vertices of polygons having interior points in the square S):

$$3n_2 \leqq 6L,$$

in consequence

$$eN < 2\frac{(s+d)^2}{a}.$$

As

$$s \geqq 3d,$$

$$(s+d)^2 \leqq \left(\frac{4}{3}s\right)^2 < 2s^2,$$

therefore

$$s^2 > \frac{(s+d)^2}{2} > \frac{eNa}{4}$$

i. e.

$$\frac{s^2}{2e} > \frac{Na}{8}.$$

Considering (3) this means that we can choose a square of area $\frac{Na}{8}$ such that the interior nodes of this square are second-rate nodes, and each of them is surrounded by three hexagons. Thus, for proving the theorem it is sufficient to show that N can exceed any number.

b) Now we prove that for a sufficiently great number s, $\frac{n_2}{n_1+n_3} > N$, where N is an arbitrary number.

In the first-rate nodes there are at least two vertices, and in the nodes surrounded by k hexagons there are k vertices; therefore in the square S there are at least $2n_1+3n_2+4n_3$ vertices. These vertices are contained by the vertices of hexagons covering the square (their number is $6L$), thus

$$2n_1+3n_2+4n_3 \leqq 6L$$

and, in consequence of (1),

$$2n_1+3n_2+4n_3 < 6\frac{(s+d)^2}{a}. \tag{4}$$

The angles of hexagons form in every first-rate node an angle π, and in every second-rate node an angle 2π. The vertices of the interior hexagons of the square S are in the interior nodes of S. Hence

$$n_1\pi+2n_2\pi+2n_3\pi > K4\pi,$$

where 4π is the sum of angles in a hexagon.

Owing to (2) we get

$$n_1+2n_2+3n_3 > 4\frac{(s-d)^2}{a}, \tag{5}$$

which can be written in the form

$$-\frac{3}{2}n_1-3n_2-3n_3 < -6\frac{(s-d)^2}{a}. \tag{6}$$

Adding the corresponding terms of inequalities (4) and (6), we have

$$\frac{n_1}{2}+n_3<\frac{24sd}{a}.$$

By this and by the inequality (5) we obtain:

$$n>2\frac{(s-d)^2}{a}-\frac{24sd}{a}=\frac{2s^2-28sd+2d^2}{a}.$$

Consequently

$$\frac{n_2}{\frac{n_1}{2}+n_3}>\frac{2s^2-2sd+2d^2}{24\,sd}=\frac{s}{12d}-\frac{7}{6}+\frac{d}{12s},$$

Therefore

$$\frac{n_2}{n_1+n_3}>\frac{s}{24d}-\frac{7}{12}+\frac{d}{24s}.$$

d being a fixed number, the value of the above expression will exceed any number if we choose s sufficiently large. This completes the proof of Theorem 1.

From this theorem it follows that, if there is a pattern with a convex hexagon, pattern can be made with the same hexagon the graph of which is isomorphic to the graph of regular tesselation with hexagons.*

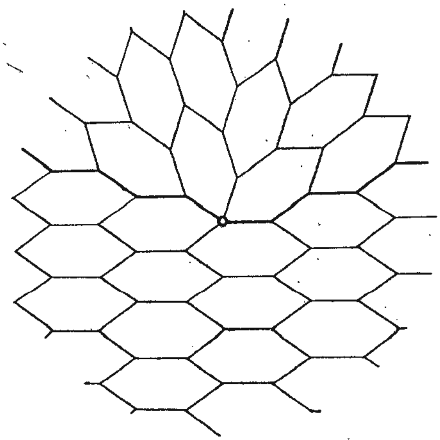

Fig. 2.

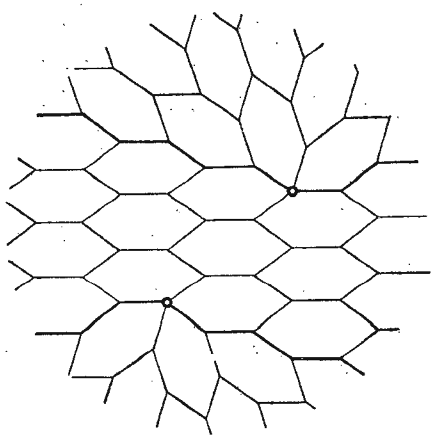

Fig. 3.

On the other hand these patterns are obviously not the only patterns with convex hexagons, as it is shown by figures 2, 3 and 4.

Using Theorem 1. we can prove the following theorems:

* From this it follows that, in the case of a given convex hexagon, to settle whether or not we can make a pattern with this hexagon, it is enough to examine the existence of a pattern the graph of which is isomorphic to the graph of the regular tesselation with hexagons.

THEOREM 2. *If there is a pattern with a convex hexagon, then at least two of the sides (of this hexagon) are equal.*

PROOF. Let the sides of hexagon $I \equiv ABCDEF$, implying the pattern, be $AB = a$, $BC = b$, $CD = c$, $DE = d$, $EF = e$ and $FA = f$, and let II and III be the hexagons joining to the edges AB and BC respectively. (Fig. 5.)

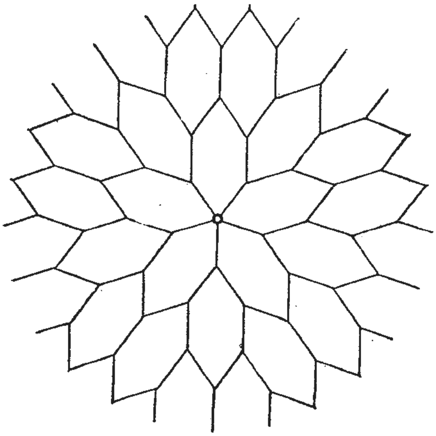

Fig. 4.

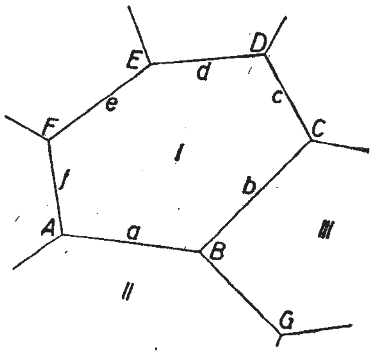

Fig. 5.

On account of theorem 1, we may suppose that the graph of this pattern which will be examined is isomorphic to the regular tesselation with hexagons; therefore the hexagons II and III are adjacent, their common edge being BG.

Supposing that the hexagon has no two equal sides, the edge AB is the side a of hexagon II, and the edge BC is the side of hexagon III. From this it follows that the edge BG in hexagon II is adjacent to a, and in hexagon III is adjacent to b. Consequently at least one of the adjacent sides of a (b or f) is equal to one the adjacent sides of b (a or c), and the hexagon has indeed two equal sides.

THEOREM 3. *If a convex hexagon has not more than two equal sides, there is a pattern with this hexagon if and only if these equal sides are opposite and parallel.*

a) The condition is necessary.

If a hexagon has only two equal sides, then there are two adjacent sides, that neither of them is equal to another side of the hexagon. Let a and b denote these sides, and let the others be c, d, e and f respectively (Fig. 5.). We may suppose that the graph of the pattern with this hexagon is the graph of Figure 6. As in the previous proof, we get that one of b and f must be equal to one of a and c. Because $a \neq b$, $a \neq f$ and $b \neq c$, this can be realized if and only if $c = f$. In this case the angles in the point B are adjacent angles of the hexagon, their sum is 2π, and consequently $c \| f$, i. e. the equal sides are opposite and parallel.

b) The condition is sufficient.

Let us reflect the hexagon to the midpoint of side a, and translate these two hexagons in such a way that side f coincides with c resp. e with f. Repeating these translations we get a ribbon, the length of which can exceed any value. Since the two broken lines forming the boundary of this ribbon are congruent with each other, we can rejoin this ribbon by translations from both sides to itself; and by repeating this we can cover an arbitrarily large part of the plane. (Fig. 6.)

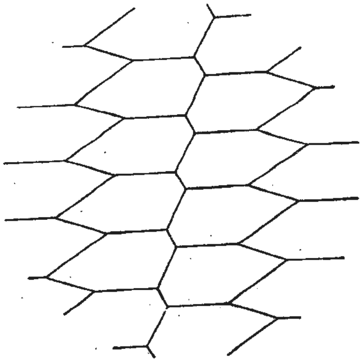

Fig. 6.

THEOREM 4. *If we can make a pattern with a hexagon, then the hexagon has three angles, the sum of which is 2π.*

PROOF: Suppose the contrary, then there is a hexagon, having no three angles with the sum 2π, suitable for making a pattern.

Let $\alpha_1, \alpha_2, \dots \alpha_6$ denote the angles of the hexagon, and let us suppose, that $\left|\alpha_1 - \frac{2\pi}{3}\right| \geqq \left|\alpha_i - \frac{2\pi}{3}\right|$, where $i = 1, 2, \dots 6$. It is obvious that $\alpha_1 \neq \frac{2\pi}{3}$, because otherwise every angle of the hexagon must be $\frac{2\pi}{3}$.

On account of theorem 1, we may consider an arbitrarily large part of the pattern, in which every node is second-rate, and three angles form an angle of 2π. It is impossible that the angles of a node should be different angles of the hexagon; consequently, two or three of them are corresponding. Taking a node, one angle of which is α_1, the angles in this node cannot be equal; hence we may suppose, that they are $\alpha_1, \alpha_1, \alpha_2$ or $\alpha_1, \alpha_2, \alpha_2$. From the first case it would follow, that $2\alpha_1 + \alpha_2 = 2\pi$, $2\left(\alpha_1 - \frac{2\pi}{3}\right) = \frac{2\pi}{3} - \alpha_2$; therefore $\left|\alpha_2 - \frac{2\pi}{3}\right| = 2\left|\alpha_1 - \frac{2\pi}{3}\right| > \left|\alpha_1 - \frac{2\pi}{3}\right|$, which is impossible. The hexagon has no angle equal to α_2, because in this case the sum of α_1, α_2 and of this angle (which naturally cannot be α_1) would be 2π. Consequently, every node of an angle α_1 is formed by two angles α_2 and by one angle α_1 only; that is to say, every domain of the chosen part of the pattern contains at least twice as many angles α_2 as α_1.

But this is impossible since it can be proved easily that in a sufficiently large part of the pattern the different angles are approximately of equal number.

This proves our theorem.

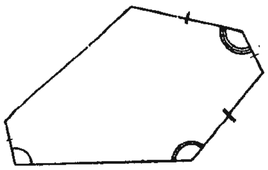

Fig. 7.

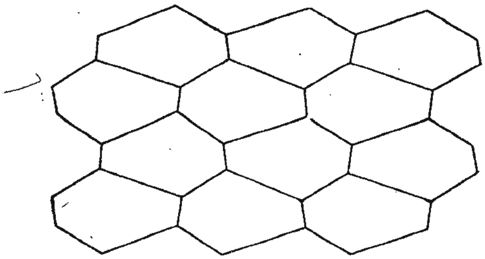

Fig. 8.

It can be proved, that if in a pattern with the hexagon $ABCDEF$ every node is second-rate, and surrounded by three hexagons, and in the nodes there are only two types of systems of angles (in one type of nodes the system $\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}$ forms the complete angle, and in the other nodes: $\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}$, where $\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}, \alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}$ are the angles of hexagon), then either $AB \# DE$ (Fig. 6.), or $FAB\sphericalangle + ABC\sphericalangle + CDE\sphericalangle = 2\pi$ with $FA = CD$ and $BC = DE$ (Fig. 7 and 8.).

It can be proved, too, as in theorems 2 and 3, that *if a hexagon has two sides, the lengths of which are different from the length of any other side, a pattern can be made with this hexagon if, and only if, the hexagon belongs to one of the above mentioned types of hexagon* (Fig. 6 and 7.).

We do not know as yet of pattern composed of another type of hexagon.

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